Solutions

3.4: Proof by Induction

In this section we will apply the proof by induction to the more general recursively defined objects in the previous section.

Question 1. Consider the pattern of numbers below. Notice that all the rows are palindromic. Will all the subsequent row be palindromic? Explain why or why not.

$$\begin{array}{c}
1 \\
1\;1 \\
1\;0\;1 \\
1\;1\;1\;1 \\
1\;0\;0\;0\;1 \\
1\;1\;0\;0\;1\;1 \\
1\;0\;1\;0\;1\;0\;1 \\
1\;1\;1\;1\;1\;1\;1\;1 \\
1\;0\;0\;0\;0\;0\;0\;0\;1 \\
1\;1\;0\;0\;0\;0\;0\;0\;1\;1 \\
1\;0\;1\;0\;0\;0\;0\;0\;1\;0\;1 \\
1\;1\;1\;1\;0\;0\;0\;0\;1\;1\;1\;1 \\
1\;0\;0\;0\;1\;0\;0\;0\;1\;0\;0\;0\;1
\end{array}$$

Yes. Suppose row n is palindromic. That is,

$$\text{row } n = \begin{cases} a_1 a_2 \cdots a_k a_k \cdots a_2 a_1 & \text{if } n = 2k \\ a_1 a_2 \cdots a_k a_{k+1} a_k \cdots a_2 a_1 & \text{if } n = 2k+1 \end{cases}$$

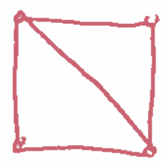
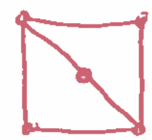
where $a_i \in \{0,1\}$ $\forall i$. Then

$$\text{row } n+1 = \begin{cases} a_1 (a_1+a_2)(a_2+a_3) \cdots (a_{k-1}+a_k) 2a_k (a_{k-1}+a_k) \cdots (a_2+a_3)(a_1+a_2) a_1 & \text{if } n=2k \\ a_1 (a_1+a_2) \cdots (a_k+a_{k+1})(a_k+a_{k+1}) \cdots (a_1+a_2) a_1 & \text{if } n = 2k+1 \end{cases}$$

and therefore row $n+1$ is palindromic.

Question 2. For an undirected graph $G = (V, E)$, let $c(G) = |V| - |E|$; i.e. the number of vertices of G minus the number of edges of G. Given a graph G, suppose you form a new graph G' by adding a vertex to G in the middle of one of the edges. Notice that $c(G) = c(G')$. Explain why adding a vertex in this manner will always preserve $c(G)$.

For example, if

$G =$  then one possibility is $G' =$ 

In this case, we can see that

$$c(G) = -1 = c(G').$$

Since adding a vertex to G increases $|V|$ by 1 and since this vertex turns one edge into two, we have $|E|$ increasing by 1.

So

$$c(G) = |V| - |E| = (|V| + 1) - (|E| + 1) = c(G').$$

To prove a statement using the principle of mathematical induction, you should think of the object in question as being defined using recursion. For a an object $R(n)$ defined for $n \geq 1$ (with base case $n = 1$), the argument to prove Statement(n) will follow as below:

Base Case. Prove Statement(1).
Recursive Case. Prove Statement($k-1$) $\Rightarrow$ Statement(k) for ~~all~~ ~~some~~ OK $k > 1$.

Statement (1) $\Rightarrow$ Statement (2) $\Rightarrow$ Statement (3) $\Rightarrow \cdots$

Theorem 1. *For any $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.* Let Statement($n$) be $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Base Case: Statement(1):

$$1 = \frac{1(1+1)}{2} \checkmark$$

Inductive Step: Suppose Statement(K-1) holds for some $K>1$. Then

$$1+2+\cdots+K-1 = \frac{(K-1)K}{2}.$$

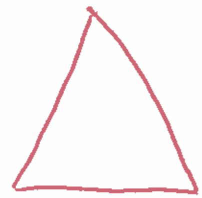
Notice $1+2+\cdots+K = (1+2+\cdots+K-1)+K$

$$= \frac{(K-1)K}{2} + K = \frac{(K-1)K+2K}{2} = \frac{K(K+1)}{2}.$$

Thus Statement(K-1) $\Rightarrow$ Statement(K) $\forall K>1$.

So Statement(n) holds for all $n \geq 1$.

Theorem 2. *Let $K(1)$, $K(2)$, $\ldots$ be the sequence of shapes whose limit is the Koch Snowflake. Prove that $K(n)$ is composed of $3 \cdot 4^{n-1}$ lines segments.*

<u>Base Case:</u> $K(1) =$ 

has $3 = 3 \cdot 4^{1-1}$ line segments ✓

<u>Inductive Case:</u> Fix $k > 1$ and suppose

$K(k-1)$ consists of $3 \cdot 4^{(k-1)-1} = 3 \cdot 4^{k-2}$ line segments.

Then each of the $3 \cdot 4^{k-2}$ line segments

is transformed into  in $K(k)$.

Therefore every line segment becomes 4 line segments, all mutually exclusive, so that $K(k)$ has

$4 \cdot (3 \cdot 4^{k-2}) = 3 \cdot 4^{k-1}$ line segments as desired.

So Statement$(k-1) \Rightarrow$ Statement(k) $\forall k > 1$

and thus

Statement(n) is true for all $n \geq 1$.

Example 1. Find a formula for $(a_1 \cdots a_n)^R \cancel{= a_n \cdots a_1}$ for any string of length $n \geq 1$.

__Claim:__ $(a_1 \cdots a_n)^R = a_n \cdots a_1$ for all strings of length $n \geq 1$.

__Base Case:__ $n=1$, $(a_1)^R = a_1$ by definition.

__Inductive Case:__ Suppose $(a_1 \cdots a_{k-1})^R = a_{k-1} \cdots a_1$ for some $k > 1$.

Then by the recursive definition, for any a_k, we have

$$
\begin{aligned}
(a_1 \cdots a_{k-1} a_k) &= a_k^R (a_1 \cdots a_{k-1})^R \\
&= a_k (a_1 \cdots a_{k-1})^R \quad \text{by Base case definition} \\
&= a_k (a_{k-1} \cdots a_1) \quad \text{by inductive hypothesis} \\
&= a_k a_{k-1} \cdots a_1
\end{aligned}
$$

as desired.

Therefore $(a_1 \cdots a_n)^R = a_n \cdots a_1$ for all $n \geq 1$.

Strong Induction. An equivalent variant of proof by induction is what is "strong induction." It is equivalent, so the adjective "strong" is a bit grandiose. With this type of induction we do the following:

Base Case. Prove Statement(1).
Recursive Case. Prove Statement(1) $\wedge \cdots \wedge$ Statement($k-1$) $\Rightarrow$ Statement(k) for all $k > 1$.

Example 2. Show that every binary tree is connected and has no simple circuits.

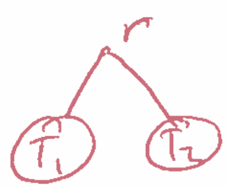
We will prove this by induction on the height of a binary tree.

Base Case: $ht(T) = 0$, this is a single vertex • which is connected w/ no simple circuits.

Inductive Hypothesis: Suppose every binary tree of height less than k is connected w/ no simple circuits for some $k > 1$

Inductive Step: Suppose T is a binary tree of height k.

Then $T =$



for some root r and two subtrees of height less than k.

By inductive hypothesis both T_1 and T_2 are connected w/ no simple circuits. So T is connected.
Furthermore, since T_1 and T_2 have no simple circuits, any simple circuit in T would have to pass through r, But then it would have to pass through r twice, a contradiction to the circuit being simple

No Homework. For practice, go through all recursively defined objects in the previous section and prove something about them inductively. See the exercises of Section 3.4 for hints on what to prove.

Group Work 1. For $n \geq 1$, prove that the n^{th} Fibonacci number is

$$F(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

Recall $F(n) = \begin{cases} 1, & \text{if } n=1 \text{ or } n=2 \\ F(n-2)+F(n-1), & \text{if } n>2. \end{cases}$

Base Cases: $n=1$, $\frac{\alpha^1-\beta^1}{\alpha-\beta} = 1$ ✓

$n=2$, $\frac{\alpha^2-\beta^2}{\alpha-\beta} = \frac{(\alpha+1)-(\beta+1)}{\alpha-\beta}$ (since α and β are the solutions to $x^2=x+1$) (✪)

$= \frac{\alpha-\beta}{\alpha-\beta} = 1$ ✓

Inductive Hypothesis: $F(i) = \frac{\alpha^i-\beta^i}{\alpha-\beta}$ for all $1 \leq i < K$ for some K.

Inductive Step: Then $F(K) = F(K-1)+F(K-2)$

$$= \frac{\alpha^{K-1}-\beta^{K-1}}{\alpha-\beta} + \frac{\alpha^{K-2}-\beta^{K-2}}{\alpha-\beta} = \frac{\alpha^{K-2}(\alpha+1)-\beta^{K-2}(\beta+1)}{\alpha-\beta}$$

$$= \frac{\alpha^{K-2}(\alpha^2)-\beta^{K-2}(\beta^2)}{\alpha-\beta} \quad \text{(by ✪)}$$

$$= \frac{\alpha^K-\beta^K}{\alpha-\beta} \quad \text{as desired.}$$

Structural Induction

Group Work 2. Define a set $X \subseteq \mathbb{Z}$ recursively as

Base Case. $4 \in X$
Recursive Case 1. If $x \in X$, then $x - 12 \in X$.
Recursive Case 2. If $x \in X$ then $x^2 \in X$.

Prove that every element of X is divisible by 4.

Base Case: 4 is divisible by 4 ✓

Inductive Hypothesis: Suppose that some $x \in X$ is divisible by 4; i.e. $x = 4a$ for some $a \in \mathbb{Z}$.

Inductive Step:

1) Since $x = 4a$, $x - 12 = 4a - 12 = 4(a-3)$ is divisible by 4

2) Since $x = 4a$, $x^2 = (4a)^2 = 4(4a^2)$ is divisible by 4.

So every element of X is divisible by 4.